Please show **all** your work! Answers without supporting work will not be given the full points. You have 3 hours to complete this exam.

Name:_____

1. Write the complex number z in polar notation (exponential form). Find its magnitude and phase:

$$z = \frac{1-i}{1+i}$$

2. Given the two sets of orthonormed basis vectors $\vec{e}_1 = (1,0)^T, \vec{e}_2 = (0,-1)^T$ and $\vec{f}_1 = (0,-1)^T, \vec{f}_2 = (-1,0)^T$ in $\mathbb{R}^2$, find the matrix of transformation $\mathbf{A}$ between them. Find its eigenvalues and eigenvectors.

3. Given the function $f(x)$ defined as

$$f(x) = \begin{cases} (1+s^2-s^4)\cos(x) & \text{if } -\theta \leq x \leq \theta \\ 0 & \text{everywhere else,} \end{cases}$$

find its mean value on the interval $[-\pi, \pi]$, assuming that s is some parameter and $|\theta| < \pi$. What value(s) of the parameter s maximize this average value of the function over the interval.

4. Find the Fourier decomposition of a rectangular periodic signal with period $T > 2$, unity height and width 2, formally defined by

$$f(x) = \begin{cases} 1 & \text{if } -1 \leq x \leq 1 \\ 0 & \text{everywhere else in } (-T/2, T/2). \end{cases}$$

5. A regular six-sided dice is tossed 10 times, resulting in the following pattern: $\{2,2,5,5,3,5,4,4,5,5\}$. What is the mean, variance and standard deviation of the sequence? Do you consider the dice to be unfair and what evidence can you propose to back up your claim?

6. Given the 1D potential with parameters U_0 and a

$$U(x) = U_0\left(1 - e^{-ax}\right)^2,$$

find an approximate solution to the differential equation

$$\frac{d^2}{dt^2}x(t) = -\nabla U$$

for $|x| << 1$ by approximating it as a harmonic oscillator around the equilibrium point $x = 0$. What would be the approximate period of the oscillations for a = 2 m^{-1} and $U_0 = 8\pi^2$ $\mathrm{m}^2/\mathrm{s}^2$.

7. A body is moving in the 3-dimensional force field $\vec{F}(\vec{x}) = \frac{\vec{x}}{(||\vec{x}||^2 - 1)}$ along the path $\vec{x}(t) = (\cos(t), \sin(t), t)$. Find the work done to push the body along this trajectory from $t = 1$ until $t = 10$. Is energy lost or gained in this motion? Is the force conservative (why or why not)?

8. A solid sphere of Aluminum with radius 1 meter is orbiting the Sun at a distance of 1 AU in a way in which half of its surface is always looking to the Sun and the other half is always looking into deep Space. What is the sphere's temperature, assuming that the Solar flux at that distance is 1367 $\mathrm{W/m}^2$, the temperature of deep Space is 2.7 K, the sphere has absorbtivity $\alpha = 0.3$ and emissivity $\epsilon = 0.7$, and that it has perfect conductivity (the entire sphere can be assumed to be at the same temperature)

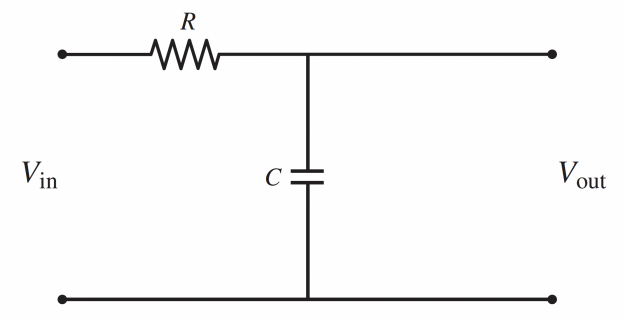
9. Three electric charges with mass m and the same charge q are held at the vertices of an equilateral triangle with side a (i.e. each charge is a distance a from each of the other two). When released, they move away from each-other due to the force of their repulsion. What is their speed when separated at distances x such that $x >> a$.

10. A coil with magnetic dipole $\mu = 1.0$ Am^2 is in a constant magnetic field with magnitude $|\vec{B}| = 10^{-4}$ T. What is the period of oscillation of the dipole around the magnetic field axis for small deflection angles $\theta_0 << \pi$ if the moment of inertia of the coil in an axis orthogonal to its dipole moment is $I_{\mu\perp} = 166$ gcm^2

11. Assume that an electromagnetic wave is transmitted by an electron gyrating (rotating in a uniform motion) around the Earth's magnetic field in an area where the magnetic field can be approximated as constant with induction $B = 25$ μT. What is the wavelenght of this wave when transmitted (above Earth's atmosphere), and what wavelength would be measured if the same wave is later detected underwater, given that the refractive index of water is $\cong 1.33$.

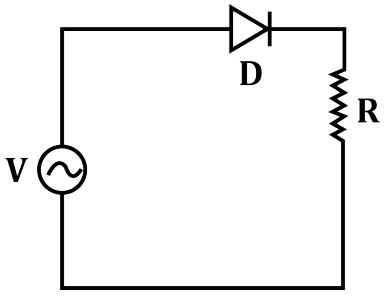
12. A cell phone's average radiated power is about 0.5 W. If the receiver at a cell tower can handle signals with peak electric fields of 1.2 mV/m, what is the maximum distance from a phone to the tower?

13. Consider the circuit shown bellow with values $R = 100\ \Omega$ and $C = 10$ nF.



If the input signal $V_{\text{in}} = \sum_{n=0}^{100} V_n \cos(\omega n t)$ is passed through it, where $\omega = 500$ kHz and $\{V_n\}_{n=0}^{100}$ are some constants, what would be the resulting output signal V_{out}.

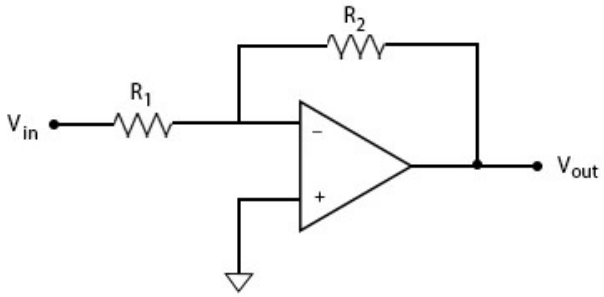
14. Consider a resistor R and a diode D connected in series to a source of alternating voltage $V = V_0 \cos(\omega t)$ for some frequency ω as shown bellow:



What is the current $I(t)$ passing through the resistor R over a full period $t \in [0, T]$, where $\omega = 2\pi/T$ (sketch or provide piecewise closed form)?

Cont.

15. The figure bellow shows an operational amplifier connected in a circuit with two resistors R_1 and R_2. What will be the output voltage V_{out} given an input voltage of $V_{\text{in}} = 5$ V? What is the application of such a circuit?



The End.